Completeness of scattering states for rough interfaces

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Abstract

Using energy conservation and causality considerations, the completeness of scattering states in established for plane waves impinging on an irregular interface. Provided certain limiting operations commute with differentiation, it is shown that surface waves need not be explicitly included in the Weyl representation of the Green's function in the presence of a rough interface. Rather surface waves are implicitly included through the poles of the scattering amplitudes. This result was used implicitly in a recently developed scheme to treat scattering in a duct using half-space scattering amplitudes.

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I Introduction

In an earlier paper the method of smoothing was applied to wave propagation in a duct with rough boundaries. The idea was to make use of plane-wave scattering amplitudes for each of the surfaces in isolation, i.e., scattering amplitudes for each surface in a half-space. Fields expressed as superpostions of scattering states associated with each surface were matched across a fictitious plane passing through the middle of the duct. Implicit in this scheme is the notion that any field in the duct below (or above) the fictious plane can be represented as a superposition of the following type:

$$\psi(\mathbf{r}) = \int d\mathbf{Q}a(\mathbf{Q})\left(\exp(i\mathbf{Q}\cdot\mathbf{R} - iqz) + \int d\mathbf{Q}'[\exp(i\mathbf{Q}'\cdot\mathbf{R} + iq'z)T(\mathbf{Q}'|\mathbf{Q})]\right). \quad (1)$$

Here $\mathbf{Q}$ represents a horizontal wave vector and $q = \sqrt{\omega^2/c^2 - \mathbf{Q}^2}$. In some instances, however, one knows there are surface waves which decay exponentially from the interface. Are these included in this formulation? In the case of an interface between two fluids, how does one incorporate waves propagating up from below the interface? The aim of the present work is to examine these questions, in effect to show under what circumstancee set of scattering states for a rough fluid-fluid interface is complete.

As indicated, the scattering states should be linear in the scattering amplitudes. One learns early on that armed with a complete set one should be able to construct a Green's function for the wave equation bilinear in the scattering states:

$$G(\mathbf{r},\mathbf{r}') \approx \int d\mathbf{k}'\psi_{\mathbf{k}'}(\mathbf{r})\psi_{\mathbf{k}'}^*(\mathbf{r}')/(\mathbf{k}^2 - \mathbf{k}'^2). \quad (2)$$

This means that the Green's function will contain terms bilinear in the scattering amplitudes. On the other hand, the Green's function in the absence of a boundary is expressible as a superposition of downgoing plane-waves below the source level (through the Weyl representation). The presence of a boundary can be accounted for by converting these downgoing waves into a sum of the downgoing wave plus reflected upgoing waves which are linear in the reflection coefficient or scattering amplitude. The Green's function in the presence of a boundary is thus the free-space Green's function plus a scattered field linear in the reflection coefficients [1]. How does one reconcile these two representations of the Green's function, one of which contains terms bilinear in the scattering amplitudes, the other of which contains terms at most linear in the scattering amplitudes? A hint comes

from the optical theorem which is a consequence of energy conservation. The optical theorem relates the imaginary part of the forward scattering amplitude (linear) to the total cross section (quadratic).

In this paper the reconcilation of these two types of expressions for the Green's function will be worked out explicitly for rough interfaces separating two fluids. How surface waves and evanescent waves come in will be shown in detail. The ideas underlying the reconciliation are energy conservation, causality, and time reversal invariance. Of course, these notions have been discussed in depth for quantum mechanical scattering by bounded objects [2]. Here the scattering surfaces are not bounded and there is the possibility that the form the wave function depends upon which side of the boundary it is to be evaluated. These features make the interface problem slightly different than the quantum problem. Additionally, role of "closed" channels must be carefully accounted for in this work. In the case of rough interfaces, (at least if the Rayleigh hypothesis holds) the wave functions can be expressed explicitly in terms of the scattering amplitudes so that one can see clearly how the connection among the general rules plays out.

Earlier work treating the completeness of elastic waves in a solid bounded by a vacuum at a smooth interface was done by Ezawa [3]. He showed completeness by close examination of the special form of the reflection coefficient and by explicitly considering the Rayleigh surface wave and evanescent waves. Here, the motivation is to show how general principles lead to completeness relations.

In section II scattering states and scattering amplitudes will be defined. General relations such as reciprocity will be reviewed, particularly the generalized optical theorem. In section III proof of completeness will be presented for rough fluid-fluid interfaces. In section IV, the Weyl representation of the Green's function for this rough interface will be discussed. The main result is that surface waves will appear explicitly in completess relations, but that in the Weyl representation of the Green's function with an interface, poles in the scattering amplitudes account for surface waves. There is an appendix containing details of the computations.

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II Scattering states

This section is concerned with solutions of the Helmholtz equation

$$(\nabla^2 + (\omega^2/c^2))\psi(\mathbf{r}) = 0 \tag{3}$$

in two half-spaces separated by a rough interface. On the upper side of the interface the sound speed is c^+ and the ambient density is ρ^+. Across the interface there is a sharp jump discontinuity in these parameters so that below the interface they become c^- and ρ^-. The field ψ represents the velocity potential for particle motions. For normal fluids presure and normal velocities must be continuous across the interface. In terms of the velocity potential these conditions are expressed by

$$
\begin{aligned}
\rho^+\psi(\mathbf{r}^+) &= \rho^-\psi(\mathbf{r}^-) \\
\mathbf{n} \cdot \nabla\psi(\mathbf{r}^+) &= \mathbf{n} \cdot \nabla\psi(\mathbf{r}^-)
\end{aligned}
\tag{4}
$$

(See Ref.[4].) In addition to boundary conditions across the interface, physical solutions of the Helmholtz equation must not diverge as $z \rightarrow \pm\infty$, however, they may exhibit exponential growth in limited domains.

Consider scattering states associated with the Helmholtz equation and these boundary conditions. For each frequency ω there are solutions (χ^+) which grow out of plane waves incident from above the interface and become a superpostion of plane waves leaving the interface both above and below the interface. Likewise there are such solutions growing out of plane waves incident from below (χ^-). Wave vectors here have horizontal components denoted by uppercase letters, e.g., $\mathbf{Q}, \mathbf{K}$... and vertical components denoted by corresponding lower case letters q, k.... (Similarly, position vectors will be denoted by upper and lower case letter, e.g. $\mathbf{r} = (\mathbf{R}, z)$). Because the plane waves must satisfy the Helmholtz equation, vertical and horizontal components of wave vectors are not independent; they must satisfy

$$q^\pm = \sqrt{\omega^2/c^{\pm 2} - \mathbf{Q}^2} \tag{5}$$

above or below the interface. The coefficients of the scattered plane waves are the scattering amplitudes, $R_{1,1}(\mathbf{Q'}|\mathbf{Q})$ for the scattering from the upper fluid (medium 1) with horizontal wave vector $\mathbf{Q}$ back into the upper fluid with horizontal wave vector $\mathbf{Q'}$, $R_{2,2}(\mathbf{Q'}|\mathbf{Q})$

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for scattering from the lower fluid (medium 2) back into the lower fluid, $T_{1,2}(\mathbf{Q'}|\mathbf{Q})$ for scattering from the lower fluid into the upper fluid and $T_{2,1}(\mathbf{Q'}|\mathbf{Q})$ for scattering from the upper into the lower fluid.

Above the highest point of the interface (H^+) and below the lowest point of the interface (H^-), scattering states can be written as follows:

$$\chi_{\mathbf{Q},q^+}^+(\mathbf{r}) = \begin{cases} \exp(i\mathbf{Q}\cdot\mathbf{R} - iq^+z) + \int d\mathbf{Q'}\exp(i\mathbf{Q'}\cdot\mathbf{R} + iq^{+\prime}z)R_{1,1}(\mathbf{Q'}|\mathbf{Q}) & \text{if } z > H^+ \\ \int d\mathbf{Q'}\exp(i\mathbf{Q'}\cdot\mathbf{R} - iq^{-\prime}z)T_{2,1}(\mathbf{Q'}|\mathbf{Q}) & \text{if } z < H^- \end{cases}$$

(6)

The scattering state χ^- is given likewise by

$$\chi_{\mathbf{Q},q^-}^-(\mathbf{r}) = \begin{cases} \int d\mathbf{Q'}\exp(i\mathbf{Q'}\cdot\mathbf{R} + iq^{+\prime}z)T_{1,2}(\mathbf{Q'}|\mathbf{Q}) & \text{if } z > H^+ \\ \exp(i\mathbf{Q}\cdot\mathbf{R} + iq^-z) + \int d\mathbf{Q'}\exp(i\mathbf{Q'}\cdot\mathbf{R} - iq^{-\prime}z)R_{2,2}(\mathbf{Q'}|\mathbf{Q}) & \text{if } z < H^- \end{cases}$$

(7)

In these equations the scattered vertical wavenumbers $q^{\pm\prime}$ are given by Eq.5 with $\mathbf{Q}$ replaced by $\mathbf{Q'}$. These solutions can be parameterized by ω rather than $q^\pm$. In order to produce scattered waves which travel away from the surface or which decay away from the surface, if q' is real, it must have the same sign as ω, and if it is pure imaginary, it must be positive imaginary. As a function of ω, $q^\pm$ has two branch points at $\pm\omega/c^\pm$. It will be convenient to take branch lines extending out from these points just under the real axis. Then q will be analytic in the upper half complex ω plane. (See Fig. 2). This is desirable because the implicit temporal phase is $\exp(-i\omega t)$. For $t < 0$ integrals over ω can be closed in the upper half plane. Other factors should be analytic there so that scattered fields vanish before the incident wave arrives at the scattering surface. (If there are surface waves, there will be poles in the upper half plane, however).

Recall that for an acoustic field with pressure p and particle velocity $\mathbf{v}$ the energy flux is [4]

$$\mathbf{f} = p\mathbf{v}.$$

(8)

For fields which represent time harmonic velocity potentials,

$$\psi = \text{Re}(\bar{\psi}\exp(-i\omega t))$$

(9)

the energy flux averaged over a period is

$$\langle \mathbf{f}\rangle = \frac{1}{T}\int_0^T \mathbf{f}dt = (\rho\omega/2)\text{Im}\bar{\psi}^*\nabla\bar{\psi}.$$

(10)

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It will be convenient for describing some of the formal properties of the scattering amplitudes to use plane waves normalized to have unit flux in the vertical direction:

$$\exp(i\mathbf{K} \cdot \mathbf{R} \pm ikz)/\sqrt{\rho\omega k/2}. \tag{11}$$

II.1   Reciprocity

Consider horizontal planes $Z^{\pm}$ above and below the rough interface separating the two fluids. Let ψ_a and ψ_b be any two solutions of the Helmholtz equation satisfying the boundary conditions, Eq.4. It follows from Green's theorem, the Helmholtz equation and the boundary conditions that

$$\int_{Z^+} d\mathbf{R}\rho^+(\psi_a\partial_z\psi_b - \psi_b\partial_z\psi_a) = \int_{Z^-} d\mathbf{R}\rho^-(\psi_a\partial_z\psi_b - \psi_b\partial_z\psi_a). \tag{12}$$

Actually this result holds even if there are a number of fluid or elastic layers between the planes Z^- and Z^+. This result also holds when the continuity of normal velocity condition is replaced by

$$\mathbf{n} \cdot \nabla\psi^+ - \mathbf{n} \cdot \nabla\psi^- = -g\rho^+\psi^+. \tag{13}$$

For flat interfaces this last boundary condition gives rise to surface waves which can be seen to manifest themselves as poles of the reflection coefficient.

One reciprocity relation follows from Eq.12 by using $\psi_a = \chi_{\mathbf{Q}}^+$ and $\psi_b = \chi_{-\mathbf{K}}^+$:

$$q^+R_{1,1}(-\mathbf{Q}| - \mathbf{K}) = k^+R_{1,1}(\mathbf{K}|\mathbf{Q}). \tag{14}$$

By using $\psi_b = \chi_{-\mathbf{K}}^-$ one finds

$$\rho^-k^-T_{2,1}(\mathbf{K}|\mathbf{Q}) = \rho^+q^+T_{1,2}(-\mathbf{Q}| - \mathbf{K}). \tag{15}$$

It follows from using $\psi_a = \chi_{\mathbf{Q}}^-$ and $\psi_b = \chi_{-\mathbf{K}}^-$ that

$$q^-R_{2,2}(-\mathbf{Q}| - \mathbf{K}) = k^-R_{2,2}(\mathbf{K}|\mathbf{Q}). \tag{16}$$

II.2   Energy conservation

It now is convenient to use flux-normalized plane waves. Define incoming plane waves by

$$\phi_{\pm,\mathbf{Q},q^\pm}^{in}(\mathbf{r}) = \exp(i\mathbf{Q} \cdot \mathbf{R} \mp iq^\pm z)/\sqrt{\rho^\pm\omega q^\pm}, \tag{17}$$

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and outgoing plane waves by

$$\phi^{out}_{\pm,\mathbf{Q},q\pm}(\mathbf{r}) = \exp(i\mathbf{Q}\cdot\mathbf{R} \pm iq^{\pm}z)/\sqrt{\rho^{\pm}\omega q^{\pm}}, \tag{18}$$

where the signs are to be chosen according to whether z is above or below the interface.

The roots in the normalization are to be understood as analytic functions with branch lines along the negative real axis. Normally the sheet that will be used will have non-negative real part. Since $\sqrt{q^*} = \sqrt{q}^*$ and since q is pure imaginary if $\omega/c < Q$, it follows that when ω is taken to be an independent variable

$$\theta(\omega/c^{\pm} - Q)\phi^{in}_{\pm,\mathbf{Q},q\pm}(\mathbf{r}) + i\theta(Q - \omega/c^{\pm})\phi^{out}_{\pm,\mathbf{Q},q\pm}(\mathbf{r}) = \phi^{out}_{\pm,-\mathbf{Q},q\pm}(\mathbf{r})^*. \tag{19}$$

In order that the scattering matrix, to be defined below, reflect reciprocity as simply as in Eq.26, it is important that the normalization described here involve q rather than $|q|$. The abbreviations

$$\theta^{\pm}(Q) = \theta(\omega/c^{\pm} - Q) \tag{20}$$

$$\bar{\theta}^{\pm}(Q) = \theta(Q - \omega/c^{\pm}), \tag{21}$$

will be used in the next section.

Suppose that above and below the interface for a limited range of z, arbitrary super-positions of incoming and outgoing flux-normalized plane waves are constructed. Such a field above the interface is

$$\psi(\mathbf{r},\omega) = \psi^{+}(\mathbf{r}) = \int d\mathbf{Q}\phi^{in}_{+,\mathbf{Q},q+}(\mathbf{r})a_{in}(\mathbf{Q}) + \phi^{out}_{+,\mathbf{Q},q+}(\mathbf{r})a_{out}(\mathbf{Q}). \tag{22}$$

Below the interface the this field is written

$$\psi(\mathbf{r},\omega) = \psi^{-}(\mathbf{r}) = \int d\mathbf{Q}\phi^{in}_{-,\mathbf{Q},q-}(\mathbf{r})b_{in}(\mathbf{Q}) + \phi^{out}_{-,\mathbf{Q},q-}(\mathbf{r})b_{out}(\mathbf{Q}). \tag{23}$$

Here the vertical wave numbers are determined from the horizontal wave numbers and ω by Eq.5. It is not claimed yet that all fields can be represented this way; that would be begging the question of completeness.

If these states obey the boundary conditions and are constructed from the scattering states $\chi^{\pm}$, it follows that the outgoing coefficients a_{out} and b_{out} are related to the incoming

coefficients a_{in} and b_{in} through the scattering amplitudes. Because normalized waves are now being used, this relationship is

$$\begin{pmatrix} a_{out}(\mathbf{K}) \\ b_{out}(\mathbf{K}) \end{pmatrix} = \int d\mathbf{Q} \begin{pmatrix} S_{1,1}(\mathbf{K}|\mathbf{Q}) & S_{1,2}(\mathbf{K}|\mathbf{Q}) \\ S_{2,1}(\mathbf{K}|\mathbf{Q}) & S_{2,2}(\mathbf{K}|\mathbf{Q}) \end{pmatrix} \begin{pmatrix} a_{in}(\mathbf{Q}) \\ b_{in}(\mathbf{Q}) \end{pmatrix}, \tag{24}$$

where the scattering matrix S, is expressed in terms of the scattering amplitudes by

$$\begin{pmatrix} S_{1,1}(\mathbf{K}|\mathbf{Q}) & S_{1,2}(\mathbf{K}|\mathbf{Q}) \\ S_{2,1}(\mathbf{K}|\mathbf{Q}) & S_{2,2}(\mathbf{K}|\mathbf{Q}) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{k^+}{q^+}}R_{1,1}(\mathbf{K}|\mathbf{Q}) & \sqrt{\frac{\rho^+k^+}{\rho^-q^-}}T_{1,2}(\mathbf{K}|\mathbf{Q}) \\ \sqrt{\frac{\rho^-k^-}{\rho^+q^+}}T_{2,1}(\mathbf{K}|\mathbf{Q}) & \sqrt{\frac{k^-}{q^-}}R_{2,2}(\mathbf{K}|\mathbf{Q}) \end{pmatrix}. \tag{25}$$

In terms of the scattering matrix S the reciprocity conditions are simply stated as

$$S(\mathbf{K}|\mathbf{Q}) = S(-\mathbf{K}|-\mathbf{Q})^T. \tag{26}$$

Note that "anti-evanescent" waves may be included in these superpositions because Q may be greater than ω/c. This is permitted if the domain in which the superposition is to represent a physical state is limited in z.

To see how energy conservation is expressed in terms of S, let $\psi_a = \psi^*$ and $\psi_b = \psi$ in Eq.12. This is permissible only if ψ^* obeys the same Helmholtz equation as ψ, i.e., $(\omega/c^\pm)^2$ must be real. In effect, there can be no dissipation.

By taking care with the cases in which vertical wave vectors $q^\pm$ are purely real or purely imaginary, one finds

$$\int_{Q<|\omega|/c^+} d\mathbf{Q}|a_{in}(\mathbf{Q})|^2 + \int_{Q<|\omega|/c^-} d\mathbf{Q}|b_{in}(\mathbf{Q})|^2 - 2\int_{Q>|\omega|/c^+} d\mathbf{Q}\,\mathrm{Im}(a_{out}(\mathbf{Q})a_{in}(\mathbf{Q})^*) =$$

$$\int_{Q<|\omega|/c^+} d\mathbf{Q}|a_{out}(\mathbf{Q})|^2 + \int_{Q<|\omega|/c^-} d\mathbf{Q}|b_{out}(\mathbf{Q})|^2 - 2\int_{Q>|\omega|/c^-} d\mathbf{Q}\,\mathrm{Im}(b_{out}(\mathbf{Q})b_{in}(\mathbf{Q})^*). \tag{27}$$

A similar result for elastic waves can be found in Ref.[[6]]. This result holds for arbitrary incident wave coefficients a_{in} and b_{in} and hence it implies constraints on the scattering matrix. To separate the cases of propagating and non-propagating waves it is convenient to define the matrix operator kernels Θ and $\bar{\Theta}$ by

$$\Theta(\mathbf{Q}|\mathbf{K}) = \begin{pmatrix} \theta(|\omega|/c^+ - Q)\delta(\mathbf{Q} - \mathbf{K}) & 0 \\ 0 & \theta(|\omega|/c^- - Q)\delta(\mathbf{Q} - \mathbf{K}) \end{pmatrix}, \tag{28}$$

and

$$\bar{\Theta}(\mathbf{Q}|\mathbf{K}) = \begin{pmatrix} \theta(Q - |\omega|/c^+)\delta(\mathbf{Q} - \mathbf{K}) & 0 \\ 0 & \theta(Q - |\omega|/c^-)\delta(\mathbf{Q} - \mathbf{K}) \end{pmatrix}. \tag{29}$$

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Now because Green's theorem holds for arbitrary incident waves, the scattering matrix must satisfy

$$\Theta = S^\dagger \Theta S + i(S^\dagger \bar{\Theta} - \bar{\Theta} S). \tag{30}$$

This is a matrix equation; matrix multiplication is implied as is integration over intermediate wavevectors. It represents a generalized optical theorem since both propagating and non-propagating waves are involved, and because there is a result here for non-forward propagation. A result of this type was used by Maystre et al [8], but incident "anti-evanescent" waves were not included by these authors. See also Ref.[7] where again exponentially growing waves were not considered.

In quantum scattering the analogous result is unitarity of the S matrix. By using the reciprocity relation, Eq.26, and the fact that the matrix operators Θ and $\bar{\Theta}$ are independent of the sign of $\mathbf{Q}$, the horzontal wavevector, it is possible to show the following alternative expression of energy conservation:

$$\Theta = S \Theta S^\dagger + i(S\bar{\Theta} - \bar{\Theta} S^\dagger). \tag{31}$$

In a space in which all waves are propagating (all channels are open) these relations reduce to the unitarity condition in quantum scattering:

$$1 = SS^\dagger = S^\dagger S \tag{32}$$

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III Completeness

Using just the scattering amplitudes it will not be possible to show completeness in a true sense. This is because the representation of scattered waves as a superposition of outgoing plane waves weighted by the scattering amplitudes might not converge as the field point is moved between the lowest (H^-) and highest points of the surface (H^+). That is to say that the Rayleigh hypothesis might be violated. Nevertheless, by using the unitarity condition, Eq.31, it is possible to discuss the following quantity for points outside the forbidden region.

$$\Delta(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^3} \int d\mathbf{Q} \left[\int_0^\infty dq^+ \chi^+_{\mathbf{Q},q^+}(\mathbf{r}) \chi^+_{\mathbf{Q},q^+}(\mathbf{r}')^* + \int_0^\infty dq^- \frac{\rho^+ c^{(-)2}}{\rho^- c^{(+)2}} \chi^-_{\mathbf{Q},q^-}(\mathbf{r}) \chi^-_{\mathbf{Q},q^-}(\mathbf{r}')^* \right]. \tag{33}$$

The vertical wavenumbers associated with the incident wave here are all real. Depending on the magnitude of the outgoing horizontal wave vectors, the scattered vertical wavevectors may be either real or pure imaginary since, for example,

$$q^{+\prime} = \sqrt{\omega^2/c^{(+)2} - \mathbf{Q}'^2} = \sqrt{\mathbf{Q}^2 + q^{(+)2} - \mathbf{Q}'^2}. \tag{34}$$

In the remainder of this section it will be shown that, in fact,

$$\Delta(\mathbf{r},\mathbf{r}') = \delta^3(\mathbf{r} - \mathbf{r}'), \tag{35}$$

which is to say that the states $\chi^\pm$ are complete in the limited sense indicated above, provided there are no surface waves.

The quantity Δ is evaluated by first expressing the scattering states χ in terms of flux-normalized plane waves ϕ and the scattering matrix S, and then changing from integration over vertical wavenumbers to integration over the frequency ω. Since the incident waves in Δ are all propagating, horizontal wavevectors $\mathbf{Q}$ are restricted to $Q < \omega/c^\pm$. In this fashion, for $z > H^+$ and $z' > H^+$, Δ can be written as

$$\Delta(\mathbf{r},\mathbf{r}') = \int_0^\infty d\omega \frac{\omega^2 \rho^+}{2(c^+)^2 (2\pi)^3}$$

$$\left\{ \int_{Q<\omega/c^+} d\mathbf{Q} \left[\phi^{in}_{+,\mathbf{Q},q^+}(\mathbf{r}) + \int d\mathbf{Q}' \phi^{out}_{+,\mathbf{Q}',q^{+\prime}}(\mathbf{r}) S_{1,1}(\mathbf{Q}'|\mathbf{Q};\omega) \right] \times \right.$$

$$\left. \left[\phi^{in}_{+,\mathbf{Q},q^+}(\mathbf{r}') + \int d\mathbf{Q}'' \phi^{out}_{+,\mathbf{Q}'',q^{+\prime\prime}}(\mathbf{r}') S_{1,1}(\mathbf{Q}''|\mathbf{Q};\omega) \right]^* + \right.$$

$$\left[\int_{Q<\omega/c^-} d\mathbf{Q}'\phi^{out}_{+,\mathbf{Q}',q+'}(\mathbf{r})S_{1,2}(\mathbf{Q}'|\mathbf{Q};\omega)\right]\left[\int d\mathbf{Q}''\phi^{out}_{+,\mathbf{Q}'',q+''}(\mathbf{r}')S_{1,2}(\mathbf{Q}''|\mathbf{Q};\omega)\right]^*\right\} \quad (36)$$

In going over to the integration over frequency, factors of $c^{(-)2}$ were canceled by

$$dq^- = \frac{\omega d\omega}{c^{(-)2}q^-}. \quad (37)$$

In the terms quadratic in the scattering matrix in Eq.36, one can recognize the (1,1) part of $S\Theta S^\dagger$ that appears in the optical theorem, Eq.31, if the integration over the incident wavevectors $\mathbf{Q}$ is performed before integration over $\mathbf{Q}'$ and $\mathbf{Q}''$. The term arising from the (1,1) matrix on the left side of the optical theorem, Eq.31 combines with the terms in Eq.36 to give $\delta(\mathbf{r}-\mathbf{r}')$. The remaining terms are linear in S, coming from the linear terms in Eq.36 and the linear terms on t'.e right side of the optical theorem. The result of using the optical theorem in Eq.36 is thus

$$\Delta(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') + \int_0^\infty d\omega \frac{\omega^2\rho^+}{2(c^+)^2(2\pi)^3} \int d\mathbf{Q}' \int d\mathbf{Q}''$$

$$\left\{\phi^{out}_{+,\mathbf{Q}',q+'}(\mathbf{r})S_{1,1}(\mathbf{Q}'|\mathbf{Q}'';\omega)\left[\theta^+(Q'')\phi^{in}_{+,\mathbf{Q}'',q+''}(\mathbf{r}')^* - i\bar{\theta}^+(Q'')\phi^{out}_{+,\mathbf{Q}'',q+''}(\mathbf{r}')^*\right] + \right.$$

$$\left.\left[\theta^+(Q')\phi^{in}_{+,\mathbf{Q}',q+'}(\mathbf{r}) + i\bar{\theta}^+(Q')\phi^{out}_{+,\mathbf{Q}',q+'}(\mathbf{r})\right]S^\dagger_{1,1}(\mathbf{Q}'|\mathbf{Q}'';\omega)\phi^{out}_{+,\mathbf{Q}'',q+''}(\mathbf{r}')^*\right\}. \quad (38)$$

Using Eq.19 for the expressions in square brackets gives

$$\Delta(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') + \int_0^\infty d\omega \frac{\omega^2\rho^+}{2(c^+)^2(2\pi)^3} \int d\mathbf{Q}' \int d\mathbf{Q}''$$

$$\left\{\phi^{out}_{+,-\mathbf{Q}',q+'}(\mathbf{r})S_{1,1}(\mathbf{Q}'|\mathbf{Q}'';\omega)\phi^{out}_{+,\mathbf{Q}'',q+''}(\mathbf{r}') + \phi^{out}_{+,\mathbf{Q}',q+'}(\mathbf{r})^*S^\dagger_{1,1}(\mathbf{Q}'|\mathbf{Q}'';\omega)\phi^{out}_{+,-\mathbf{Q}'',q+''}(\mathbf{r}')^*\right\}. \quad (39)$$

Because reciprocity implies

$$S^\dagger_{1,1}(\mathbf{Q}'|\mathbf{Q}'';\omega) = S^*_{1,1}(\mathbf{Q}''|\mathbf{Q}';\omega) = S^*_{1,1}(-\mathbf{Q}'|-\mathbf{Q}'';\omega), \quad (40)$$

the two terms in braces are complex conjugates.

At this point is is useful to switch back to unnormalized plane waves and to replace $S_{1,1}$ by $R_{1,1}$. Because of the way branch cuts have been taken,

$$q(-\omega) = -q^*(\omega),$$

and because $\chi_Q(\mathbf{r};\omega)$ and $\chi_{-Q}(\mathbf{r};-\omega)^*$ grow out of the same incident plane wave, one concludes that

$$R_{1,1}(\mathbf{Q}'|\mathbf{Q}'';\omega) = R_{1,1}(-\mathbf{Q}'|-\mathbf{Q}'';-\omega)^*. \tag{41}$$

Changing signs of the dummy variables $\mathbf{Q}'$ and $\mathbf{Q}''$ for the second term in braces and letting $\omega \to -\omega$ for this term gives

$$\Delta(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') + \int_{-\infty}^{\infty} d\omega \frac{\omega}{(c^+)^2(2\pi)^3} \int d\mathbf{Q}' \int d\mathbf{Q}''$$

$$\exp(-i\mathbf{Q}'\cdot\mathbf{R} + i\mathbf{Q}''\cdot\mathbf{R}' + iq^{+\prime}z + iq^{+\prime\prime}z')R_{1,1}(\mathbf{Q}'|\mathbf{Q}'';\omega)/q^{+\prime\prime}. \tag{42}$$

Anticipating the next section, one might identify the integral above to be the time derivative (because of the factor of ω) of the scattered field arising from a point source at $\mathbf{r}'$ originating at time $t = 0$ and evaluated at $\mathbf{r}$ and time $t = 0$. This is what one would guess from the Weyl representation of the free-space Green's function. Since there should normally be no scattered field until a time later than the origin of the incident field, the integral in Eq.42 should vanish. However, if there are surface waves, these can exist without excitation from above. In that case the "scattered" field might not vanish at $t = 0$. The existence of surface waves and the vanishing of the integral in Eq.42 hinges on the analytic properties of $R_{1,1}$ as a function of frequency, ω. Because in the present case z and z' are positive and the vertical wave vectors q have non-negative imaginary parts in the upper half complex ω plane, one should be able to add a large semi-circle in the upper half complex ω plane to the contour of integration in Eq.42 and evaluate the entire integral using the residues of R, assuming that it is possible to perform the integration over ω before the integration over $\mathbf{Q}'$ and $\mathbf{Q}''$.

In the case of a flat fluid-fluid interface, the scattering amplitude is given by

$$R_{1,1}(\mathbf{Q}|\mathbf{Q}';\omega) = \delta^2(\mathbf{Q}-\mathbf{Q}')\frac{q^+\rho^- - q^-\rho^+}{q^+\rho^- + q^-\rho^+}. \tag{43}$$

The reflection coefficient is analytic in the upper half ω plane so that the frequency integral in Eq.42 vanishes. (Recall that branch cuts extend outward from $\omega + i\epsilon = \pm c^\pm Q$ just below the real axis.) In this case, the integral on the right side of Eq.42 vanishes, showing that the scattering states $\chi^\pm$ are complete.

If, on the other hand, the boundary condition Eq.13 is used instead of continuity of the normal derivatives, and if $c^+ = c^-$ and $\rho^+ = \rho^-$, then the scattering amplitude for a

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flat surface is given by

$$R_{1,1}(\mathbf{Q}|\mathbf{Q}';\omega) = \delta^2(\mathbf{Q} - \mathbf{Q}')\frac{ig/2}{q - ig/2}. \tag{44}$$

There is now a pole in the upper half ω-plane at

$$\omega = ic\sqrt{g^2/4 - Q^2}, \tag{45}$$

provided $Q < g/2$. In this case, the integral on the right hand side of Eq.42 becomes

$$\frac{-1}{(2\pi)^2} \int_{Q<g/2} d\mathbf{Q} \frac{g}{2} \exp(-i\mathbf{Q} \cdot (\mathbf{R} - \mathbf{R}') - g(z + z')/2). \tag{46}$$

This can be recognized as the contribution of normalized surface waves

$$\chi_{\mathbf{Q}}^{S}(\mathbf{r}) = \exp(-i\mathbf{Q} \cdot \mathbf{R} - g|z|/2)\sqrt{\frac{g}{2}} \tag{47}$$

to the completeness integrals.

To complete the demonstration of the relationship between $\Delta(\mathbf{r},\mathbf{r}')$ and $\delta(\mathbf{r} - \mathbf{r}')$, $\Delta(\mathbf{r},\mathbf{r}')$ needs to be evaluated for $z > H^+$ and $z' < H^-$. This is done in the Appendix. (Other cases of z and z' follow from these two just by exchanging the indices labeling the two media.) The result is that for $z > H^+$ and $z < H^-$, Δ becomes

$$\Delta(\mathbf{r},\mathbf{r}') = \int_{-\infty}^{\infty} d\omega \frac{\omega}{(c^+)^2(2\pi)^3} \int d\mathbf{Q}' \int d\mathbf{Q}''$$

$$\frac{\rho^+}{\rho^-} \exp(-i\mathbf{Q}' \cdot \mathbf{R} + i\mathbf{Q}'' \cdot \mathbf{R}' + iq^{+\prime}z - iq^{-\prime\prime}z')T_{1,2}(\mathbf{Q}'|\mathbf{Q}'';\omega)/q^{-\prime\prime}. \tag{48}$$

This integral again can be recognized as the time derivative at $t = 0$ of the scattered field evaluated at $\mathbf{r}$, above the interface arising from a point source located below the interface at $\mathbf{r}'$. If the scattering amplitude T is analytic in the upper half ω plane, the integral vanishes. Again this is consistent with the requirement of causality when there are not surface waves.

Thus if the scattering amplitudes are analytic in the upper half plane,

$$\Delta(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$

and the the scattering states $\chi^{\pm}$ are complete. If so the scattering states here are the normal modes described by Tolstoy [9]. Again note that in the the construction of Δ incident

waves are all propagating, while scattered waves may be propagating or evanescent. Also, it should be said that although the flat surface reflection and transmission coefficients might be analytic in the upper half ω plane, this is no guarantee that the corresponding rough interface scattering amplitudes are analytic there. Roughness may induce surface waves, at least for the mean field (See [5]).

IV Green's functions

In this section, the results of the last section will be used to construct the Green's function for the rough interface problem at least outside the region between H^- and H^+. Define Δ_G similarly to Δ:

$$\Delta_G(\mathbf{r},\mathbf{r}';\lambda) = \frac{1}{(2\pi)^3} \int d\mathbf{Q} \left[\int_0^\infty \frac{dq^+ c^{(+)2}}{(\lambda^2 - \omega^{(+)2})} \chi_{\mathbf{Q},q^+}^+(\mathbf{r}) \chi_{\mathbf{Q},q^+}^+(\mathbf{r}')^* + \right.$$

$$\left. \int_0^\infty \frac{dq^- c^{(+)2}}{(\lambda^2 - \omega^{(-)2})} \frac{\rho^+ c^{(-)2}}{\rho^- c^{(+)2}} \chi_{\mathbf{Q},q^-}^-(\mathbf{r}) \chi_{\mathbf{Q},q^-}^-(\mathbf{r}')^* \right], \tag{49}$$

where

$$\omega^{(\pm)2} = c^{(\pm)2}(Q^2 + q^{(\pm)2}). \tag{50}$$

The Green's function, $G(\mathbf{r},\mathbf{r}')$ should satisfy the same boundary conditions across the interface as the scattering states $\chi^\pm$ and in addition,

$$(\nabla^2 + \lambda^2/c^{(\pm)2})G(\mathbf{r},\mathbf{r}';\lambda) = \delta(\mathbf{r} - \mathbf{r}'). \tag{51}$$

The function Δ_G will be examined as a candidate for G. It is clear that Δ_G does satisfy the boundary conditions across the interface. However, it is also clear that if the Helmholtz operator can be passed through the various integrations in G, then

$$(\nabla^2 + \lambda^2/c^{(\pm)2})\Delta_G(\mathbf{r},\mathbf{r}';\lambda) = \Delta(\mathbf{r},\mathbf{r}';)$$

$$= \delta(\mathbf{r} - \mathbf{r}') - \int_{-\infty}^\infty \frac{\omega d\omega}{2\pi i c^{(+)2}} \psi(\mathbf{r},\mathbf{r}';\omega). \tag{52}$$

The function ψ is an abbreviation for the double integral appearing in Eq.42. When $z > H^+$ and $z' > H^+$ then

$$\psi(\mathbf{r}|\mathbf{r}';\omega) = -\frac{2\pi i}{(2\pi)^3} \int d\mathbf{Q}' \int d\mathbf{Q}''$$

$$\exp(-i\mathbf{Q}' \cdot \mathbf{R} + i\mathbf{Q}'' \cdot \mathbf{R}' + iq^{+\prime}z + iq^{+\prime\prime}z')R_{1,1}(\mathbf{Q}'|\mathbf{Q}'';\omega)/q^{+\prime\prime}. \tag{53}$$

When $z > H^+$ and $z' < H^+$, ψ is read out from Eq.48: $R_{1,1}$ is replaced by $\rho^+ T_{1,2}/\rho^-$ and $q^{+\prime\prime}$ is replaced by $-q^{-+\prime\prime}$. Note that reciprocity implies that

$$\psi(\mathbf{r}|\mathbf{r}';\omega) = \psi(\mathbf{r}'|\mathbf{r};\omega) \tag{54}$$

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Evaluation of Δ_G can be carried out by the same steps used to evaluate Δ, keeping track of terms quadratic in the scattering amplitudes and applying the optical theorem, Eq. 31. The result of these manipulations is that Δ_G becomes

$$\Delta_G(\mathbf{r},\mathbf{r}';\lambda) = \int_{-\infty}^{\infty} \frac{\omega d\omega}{(\lambda^2 - \omega^2)}$$

$$\left\{\int_{Q<\omega/c^+} \frac{d\mathbf{Q}}{(2\pi)^3} \exp\left[i\mathbf{Q}(\mathbf{R}-\mathbf{R}') + iq^+(z-z')\right]/q^+ - \frac{1}{2\pi i}\psi(\mathbf{r}|\mathbf{r}';\omega)\right\}. \qquad (55)$$

By letting $\lambda \to \lambda + i\epsilon$ and adding a semi-circle to the contour in the upper or lower half ω plane, depending on the sign of $z - z'$, the first integral is seen to be the Weyl representation of the free-space Green's function with frequency λ:

$$G_{free}(\mathbf{r}|\mathbf{r}';\lambda) = \int \frac{d\mathbf{Q}}{(2\pi)^2 2iq^+(\lambda)} \exp\left[i\mathbf{Q}(\mathbf{R}-\mathbf{R}') + iq(\lambda)^+|z-z'|\right]. \qquad (56)$$

In the second integral, involving ψ, the contour can be closed in the upper half ω plane when both z and z' are positive. There are two kinds of contributions. First there is the residue from the pole at $\omega = \lambda + i\epsilon$ and second there are the contributions from the branch cuts or poles of ψ. As a result, Δ_G can be written as

$$\Delta_G(\mathbf{r}|\mathbf{r}';\lambda) = G_{free}(\mathbf{r}|\mathbf{r}';\lambda) + \psi(\mathbf{r}|\mathbf{r}';\lambda)/2 - \int_B \frac{\omega d\omega}{(\lambda^2 - \omega^2)(2\pi i)}\psi(\mathbf{r}|\mathbf{r}';\omega). \qquad (57)$$

The contour B encloses the singularities and branch cuts of ψ in the upper half plane but not $\lambda + i\epsilon$. When acted upon by the Helmholtz operator, the last integral yields the integral on the right side of Eq.52. Thus if this integral is subtracted from Δ_G, the result is the Green's function:

$$G(\mathbf{r},\mathbf{r}';\lambda) = G_{free}(\mathbf{r}|\mathbf{r}';\lambda) + \psi(\mathbf{r}|\mathbf{r}';\lambda)/2 \qquad (58)$$

One can check easily enough to make sure that this satisfies proper boundary conditions across the interface, if this construction is repeated for z' below the surface, and if the Rayleigh hypothesis holds. The point of this exercise is that the poles and branch cuts of the scattering amplitudes do not make any explicit contribution to the Green's function. By adding ψ to the free space Green's function, the incoming plane waves of the Weyl representation are simply replaced by the scattering states. Surface waves are automatically accounted for even though surface waves are explicitly required for completeness.

Acknowledgements

Appendix

In this appendix, $\Delta(\mathbf{r}, \mathbf{r}')$ is evaluated for $z > H^+$ and $z' < H^-$. Some the the details sketched in the text for the case $z' > H^+$ will be made more explicit here. For example, in terms of normalized plane waves the scattering states are given by

$$
\chi^+_{\mathbf{Q},q^+}(\mathbf{r}) = \begin{cases} \sqrt{\frac{\rho^+\omega q^+}{2}}[\phi^{in}_{+,\mathbf{Q},q^+}(\mathbf{r}) \;+\; \int d\mathbf{Q}'\phi^{out}_{+,\mathbf{Q}',q^+,}S_{1,1}(\mathbf{Q}'|\mathbf{Q};\omega)] & z > H^+ \\[3mm] \sqrt{\frac{\rho^+\omega q^+}{2}}\int d\mathbf{Q}'\phi^{out}_{-,\mathbf{Q}',q^-,}S_{2,1}(\mathbf{Q}'|\mathbf{Q};\omega) & z < H^- \end{cases}
$$

(A1)

$$
\chi^-_{\mathbf{Q},q^-}(\mathbf{r}) = \begin{cases} \sqrt{\frac{\rho^-\omega q^-}{2}}[\phi^{in}_{-,\mathbf{Q},q^-}(\mathbf{r}) \;+\; \int d\mathbf{Q}'\phi^{out}_{-,\mathbf{Q}',q^-,}S_{2,2}(\mathbf{Q}'|\mathbf{Q};\omega)] & z < H^- \\[3mm] \sqrt{\frac{\rho^-\omega q^-}{2}}\int d\mathbf{Q}'\phi^{out}_{+,\mathbf{Q}',q^+,}(\mathbf{r})S_{1,2}(\mathbf{Q}'|\mathbf{Q};\omega) & z > H^+ \end{cases}
$$

(A2)

It follows from the definition of Δ, Eq. 33 that

$$
\Delta(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^3}\int d\mathbf{Q}\frac{\rho^+\omega q^+}{2}\int_0^\infty dq^+\left[\phi^{in}_{+,\mathbf{Q},q^+}(\mathbf{r}) + \int d\mathbf{Q}'\phi^{out}_{+,\mathbf{Q}',q^+,}(\mathbf{r})S_{1,1}(\mathbf{Q}'|\mathbf{Q};\omega)\right] \times
$$

$$
\left[\int d\mathbf{Q}''\phi^{out}_{-,\mathbf{Q}'',q^-,}(\mathbf{r}')S_{2,1}(\mathbf{Q}''|\mathbf{Q};\omega)\right]^* +
$$

$$
\frac{1}{(2\pi)^3}\int d\mathbf{Q}\int_0^\infty dq^-\frac{\rho^-\omega q^-}{2}\frac{\rho^+(c^-)^2}{\rho^-(c^+)^2}\left[\int d\mathbf{Q}'\phi^{out}_{+,\mathbf{Q}',q^+,}(\mathbf{r})S_{1,2}(\mathbf{Q}'|\mathbf{Q};\omega)\right] \times
$$

$$
\left[\phi^{in}_{-,\mathbf{Q},q^-}(\mathbf{r}) + \int d\mathbf{Q}''\phi^{out}_{-,\mathbf{Q}'',q^-,}(\mathbf{r}')S_{2,2}(\mathbf{Q}''|\mathbf{Q};\omega)\right]^* \qquad \text{(A3)}
$$

Now replace the variables of integration $q^\pm$ by ω using

$$
q^\pm = \sqrt{\omega^2/(c^\pm)^2 - Q^2} \qquad \text{(A4)}
$$

and change the order of integration so that the integration over $Q < \omega/c^\pm$ is first, followed by integration over $\mathbf{Q}'$ and $\mathbf{Q}''$ and then ω. Multiplying out terms in S gives

$$
\Delta(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^3}\int_0^\infty d\omega\frac{\omega^2\rho^+}{2(c^\pm)^2}\left\{\int_{Q<\omega/c^+}d\mathbf{Q}\int d\mathbf{Q}''\phi^{in}_{+,\mathbf{Q},q^+}(\mathbf{r})S^\dagger_{1,2}(\mathbf{Q}|\mathbf{Q}'';\omega)\phi^{out}_{-,\mathbf{Q}'',q^-,}(\mathbf{r}')^* + \right.
$$

$$
\int_{Q<\omega/c^-}d\mathbf{Q}\int d\mathbf{Q}'\phi^{out}_{+,\mathbf{Q}',q^+}(\mathbf{r})S_{1,2}(\mathbf{Q}'|\mathbf{Q};\omega)\phi^{in}_{-,\mathbf{Q}',q^-}(\mathbf{r}')^* +
$$

$$
\int d\mathbf{Q}'d\mathbf{Q}''\left[\int_{Q<\omega/c^+}\phi^{out}_{+,\mathbf{Q}',q'^+}(\mathbf{r})S_{1,1}(\mathbf{Q}'|\mathbf{Q};\omega)S^\dagger_{1,2}(\mathbf{Q}|\mathbf{Q}'';\omega)\phi^{out}_{-,\mathbf{Q}'',q^-,}(\mathbf{r}')^* + \right.
$$

$$\int_{Q<\omega/c-} \phi^{out}_{+,\mathbf{Q}',q'+}(\mathbf{r})S_{1,2}(\mathbf{Q}'|\mathbf{Q};\omega)S^{\dagger}_{2,2}(\mathbf{Q}|\mathbf{Q}'';\omega)\phi^{out}_{-,\mathbf{Q}'',q-''}(\mathbf{r}')^{*}\Big]\Big\}. \tag{A5}$$

The integrals over $\mathbf{Q}$ in the last expression are equivalent to the $\mathbf{Q}',\mathbf{Q}''$ elements of

$$S_{1,1}\Theta^{+}(Q)S^{\dagger}_{1,2} + S_{1,2}\bar{\Theta}^{+}(Q)S_{2,2}.$$

This is part of the $(1,2)$ matrix in the optical theorem, Eq.31 which implies

$$\int_{Q<\omega/c+} S_{1,1}(\mathbf{Q}'|\mathbf{Q};\omega)S^{\dagger}_{1,2}(\mathbf{Q}|\mathbf{Q}'';\omega) + \int_{Q<\omega/c-} S_{1,2}(\mathbf{Q}'|\mathbf{Q};\omega)S^{\dagger}_{2,2}(\mathbf{Q}|\mathbf{Q}'';\omega) =$$
$$- i\left[S_{1,2}(\mathbf{Q}'|\mathbf{Q}'';\omega)\bar{\Theta}^{-}(Q'') - \bar{\Theta}^{+}(Q')S^{\dagger}_{1,2}(\mathbf{Q}'|\mathbf{Q}'';\omega)\right]. \tag{A6}$$

Substituting this result into the last expression for Δ eliminates the terms bilinear in S. The remaining terms linear in S can be written

$$\Delta(\mathbf{r},\mathbf{r}') = \int_{0}^{\infty} d\omega \frac{\omega^{2}\rho^{+}}{2(c^{+})^{2}(2\pi)^{3}} \int d\mathbf{Q}' \int d\mathbf{Q}''$$

$$\left\{\phi^{out}_{+,\mathbf{Q}',q+'}(\mathbf{r})S_{1,2}(\mathbf{Q}'|\mathbf{Q}'';\omega)\left[\theta^{-}(Q'')\phi^{in}_{-,\mathbf{Q}'',q-''}(\mathbf{r}')^{*} - i\bar{\theta}^{-}(Q'')\phi^{out}_{-,\mathbf{Q}'',q-''}(\mathbf{r}')^{*}\right] +\right.$$

$$\left.\left[\theta^{+}(Q')\phi^{in}_{+,\mathbf{Q}',q+'}(\mathbf{r}) + i\bar{\theta}^{+}(Q')\phi^{out}_{+,\mathbf{Q}',q+'}(\mathbf{r})\right]S^{\dagger}_{1,2}(\mathbf{Q}'|\mathbf{Q}'';\omega)\phi^{out}_{-,\mathbf{Q}'',q-''}(\mathbf{r}')^{*}\right\}. \tag{A7}$$

Usinq Eq.19 to replace the terms in square brackets gives

$$\Delta(\mathbf{r},\mathbf{r}') = \int_{0}^{\infty} d\omega \frac{\omega^{2}\rho^{+}}{2(c^{+})^{2}(2\pi)^{3}} \int d\mathbf{Q}' \int d\mathbf{Q}''$$

$$\left\{\phi^{out}_{+,\mathbf{Q}',q+'}(\mathbf{r})S_{1,2}(\mathbf{Q}'|\mathbf{Q}'';\omega)\phi^{out}_{-,-\mathbf{Q}'',q-''}(\mathbf{r}')+\right.$$

$$\left.\phi^{out}_{+,-\mathbf{Q}',q+'}(\mathbf{r})^{*}S^{\dagger}_{1,2}(\mathbf{Q}'|\mathbf{Q}'';\omega)\phi^{out}_{-,\mathbf{Q}'',q-''}(\mathbf{r}')^{*}\right\}. \tag{A8}$$

The reciprocity relation of Eq.40 is for transmission expressed by

$$S^{\dagger}_{1,2}(\mathbf{Q}'|\mathbf{Q}'';\omega) = S^{*}_{2,1}(\mathbf{Q}''|\mathbf{Q}';\omega) = S^{*}_{1,2}(-\mathbf{Q}'|-\mathbf{Q}'';\omega). \tag{A9}$$

Because

$$\exp(-i\mathbf{Q}\cdot\mathbf{R} - iq^{-}(\omega)z)^{*} = \exp(i\mathbf{Q}\cdot\mathbf{R} + iq^{-}(-\omega)z), \tag{A10}$$

it follows that

$$T_{1,2}(\mathbf{Q}'|\mathbf{Q};-\omega) = T^{*}_{1,2}(-\mathbf{Q}'|-\mathbf{Q};-\omega), \tag{A11}$$

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and therefore

$$S_{1,2}(\mathbf{Q'}|\mathbf{Q''};\omega) = S_{1,2}^*(-\mathbf{Q'}|-\mathbf{Q''};-\omega). \tag{A12}$$

With this result the two terms in Δ can be combined to give a single integral over frequency from $-\infty$ to $+\infty$. Expressing the result in unnormalized plane waves and $T_{1,2}$ finally gives

$$\Delta(\mathbf{r},\mathbf{r'}) = \int_0^\infty d\omega \frac{\omega\rho^+}{\rho^-(c^+)^2(2\pi)^3} \int d\mathbf{Q'} \int d\mathbf{Q''}$$

$$\exp(i(\mathbf{Q'}\cdot\mathbf{R} - \mathbf{Q''}\cdot\mathbf{R'}) + iq^{-\prime}z - iq^{-\prime\prime}z')T_{1,2}(\mathbf{Q'}|\mathbf{Q''};\omega)/q^{-\prime\prime}(\omega). \tag{A13}$$
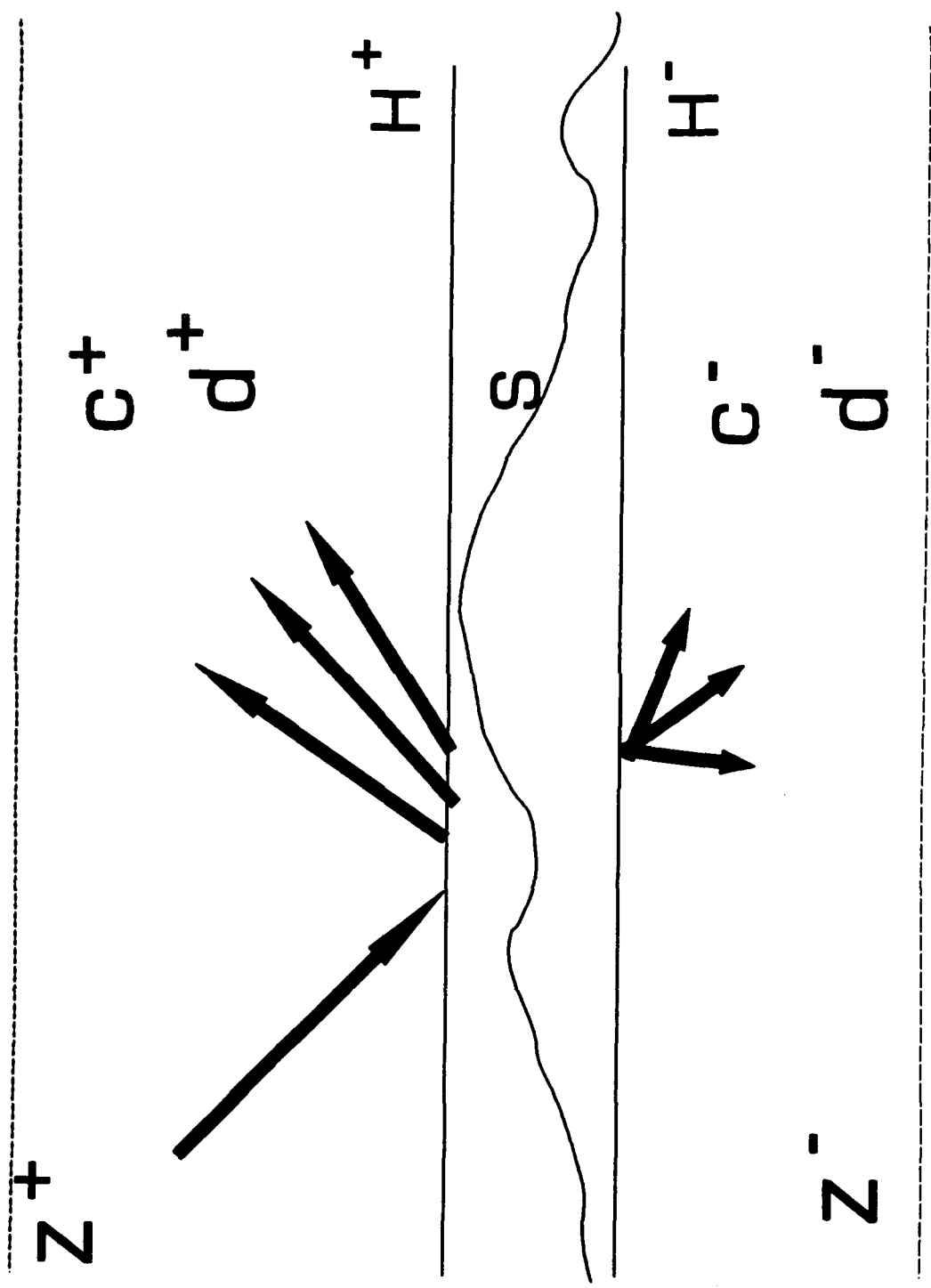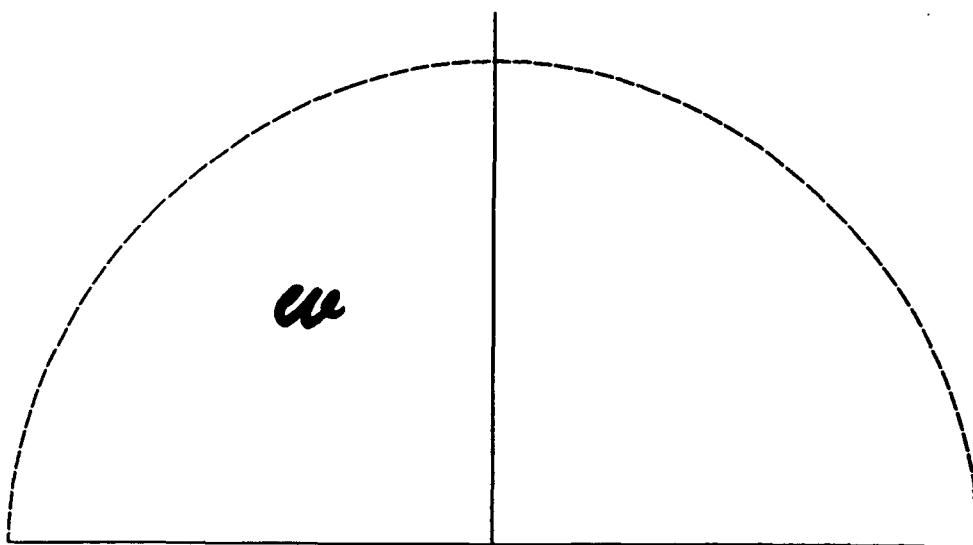
This is Eq.48 of the text.

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Figure Captions

Figure 1: The geometry of scattering from a rough interface. The rough interface separating fluids of density $\rho^+ = d^+$ and sound speed c^+ on the top and density $\rho^- = d^-$ and sound speed c^- on the bottom. The interface is bounded between parallel planes at $z = H^\pm$. Wave can be incident from either side of the interface.

Figure 2: Contours of integration in the complex ω plane. The dotted half-lines indicate the branch cuts associated with $q^\pm$. The dashed line and the real axis is the actual contour of integration closed by a semi-circle at infinity.